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IMPROVEMENT IN MODELLING AND PREDICTING SOME PREDATOR-PREY POPULATIONS

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Abstract

In this paper, we are interested in the estimation and the prediction problems for a cyclic predator-prey populations described by the Lotka-Volterra ODE system, for which the parameters of the ODE are some functions of time. To this end, we view the actual population sizes as random perturbations of the solutions to the ODE system, where these perturbations follow correlated Ornstein-Uhlenbeck processes. By using certain re-parameterization result, we show how to estimate the parameters and how to predict the population sizes. Finally, we analyze two data sets in order to illustrate the application of the suggested approach.

1. Introduction

Over the years, ecologists have observed that an oscillatory behaviour is what one observes in many animal populations and an extensive literature has evolved on the subject. To give some references, we quote Kendall et al. [11], Ginzburg and Taneyhill [10], Royama [17], Renshaw [16], Brillinger [2], and Boyce [1] among others. In Froda and Nkurunziza

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[7], the authors consider the predator-prey interacting species as described by the classical Lotka-Volterra ODE system (see Lotka [13] or Volterra [19]),

$$\dot{u}(t) = (\alpha - \beta v(t))u(t), \quad \dot{v}(t) = (\gamma u(t) - \delta)v(t), \quad (u(0), v(0)) = (x_0, y_0), \tag{1}$$

where $x_0 > 0, y_0 > 0,$ and $(\alpha > 0, \beta > 0, \gamma > 0, \delta > 0)$. As interpretation, $u(t)$ represents the size of the prey population, $\dot{u}(t)$ its derivative with respect to time, and $v(t)$ represents the size of the predator population, $\dot{v}(t)$ its derivative with respect to time, the parameter δ is the death rate of the predator, when the prey is absent, α is the birth rate of the prey, when the predator is absent, and the parameters γ and β express the interaction between the two populations.

In order to take care of the oscillatory behaviour fact and the roughness observed in real data, Froda and Nkurunziza [7] suggested a stochastic model in continuous time, in which the observed population sizes are viewed as the deterministic solution of the ODE (1) plus correlated Ornstein-Uhlenbeck processes.

In this paper, we consider a version of the model (1), where each parameter is multiplied by a certain positive and continuous functions of time t. Namely, from (1), we have

$$\dot{x}(t) = \kappa(t)(\alpha - \beta y(t))x(t), \quad \dot{y}(t) = \kappa(t)(\gamma x(t) - \delta)y(t), \quad (x(0), y(0)) = (x_0, y_0), \tag{2}$$

where $\kappa(t)$ is a strictly positive and continuous function of time t. Note that the ODE (1) is a particular case of the ODE (2) with $\kappa(t) = 1,$ for all $t \geqslant 0$.

Further, let $\{(X(t), Y(t)), t \geqslant 0\}$ be the observed population sizes process, where $\{X(t), t \geq 0\}$ is the size of the prey population, and $\{Y(t), t \geq 0\}$ is the size of the predator population. These observed population sizes, $(X(t), Y(t)),$ are viewed as the deterministic solution of (2), $(x(t), y(t))$ plus an error, which follows a pair of two correlated Ornstein-Uhlenbeck processes. From a statistical point of view, we are interested in predicting the population sizes at time t, based on the past observed values of the process $\{(X(s), Y(s)), 0 \leqslant s < t\}$.

In Section 3, we establish a certain re-parameterization result (Corollary 3.2 or Corollary 3.3) that shows that the solution of (2) $(x(t), y(t))$ stays on the same closed curve as $(u(t), v(t))$, that is, the solution of the ODE (1). The estimation method is based on the fact that the ODE system (2) admits closed orbits.

More precisely, there exists a constant η such that any solution $(x(t), y(t))$ stays on the closed curve $H(x(t), y(t)) = H(u(t), v(t)) = \eta$ with

$$H(u(t), v(t)) = (\gamma u(t) - \delta \log u(t)) + (\beta v(t) - \alpha \log v(t)), \quad \forall t \geqslant 0. \tag{3}$$

As in Froda and Nkurunziza [7], we assume that

$$\log X(t) = \log x(t) + e_t^X, \qquad \log Y(t) = \log y(t) + e_t^Y, \tag{4}$$

where $(x(t), y(t))$ are solutions to (2) and e_t^X, e_t^Y, $t \geqslant 0$, are two correlated Ornstein-Uhlenbeck processes as described in Section 2. For more detail about this stochastic model, the reader is referred to Froda and Nkurunziza [7].

First, by using the model (4), we derive the estimate of the values of the four parameters $\gamma, \beta, \delta, \alpha$. Further, by using the model (4), we show how to predict future values of the process $(X(t), Y(t)), t \geqslant 0$. Methodologically, we establish (in Section 3) a certain re-parameterization result that allows us to transform the estimation problem based on ODE (2) to the estimation problem based on ODE (1) and then, the rest of the framework becomes the same as that given in Froda and Nkurunziza [7].

The remainder of the paper is organized as follows. Section 2 gives the statistical model and preliminary results. Section 3 gives some qualitative properties of ODE and the main idea of our estimation and prediction method. In Section 4, we present the main steps of general framework of estimation and prediction. Section 5 is asymptotic results. Continuing Section 6 illustrates our method as applied to the Canadian mink-muskrat data as well as to the paramecium-didinium data sets. Finally, Section 7 is conclusion and, technical proofs are given in the Appendix.

2. Statistical Model and Preliminary Results

As given in (4), we consider $\log X(t) = \log x(t) + e_t^X$, $\log Y(t) = \log y(t) + e_t^Y$, where $(x(t), y(t))$ is a solution to (2), and each noise component $\{(e_t^X, e_t^Y), t \geqslant 0\}$ is an Ornstein-Uhlenbeck process (see Kutoyants [12], p. 51), with a particular dependence structure defined below. More precisely, we have,

$$de_t^X = -ce_t^X dt + \tau\, dW_t^X, \quad de_t^Y = -ce_t^Y dt + \tau\, dW_t^Y, \quad \tau > 0, \quad c > 0, \tag{5}$$

where $\{W_t^X, t \geqslant 0\}$ and $\{W_t^Y, t \geqslant 0\}$ are Wiener processes, which satisfy the following Assumption $(\mathcal{C}_1)$.

Assumption $(\mathcal{C}_1)$**.** The Wiener processes $\{W_t^X, t \geqslant 0\}$ and $\{W_t^Y, t \geqslant 0\}$ are jointly Gaussian and such that, for all $i, j = 1, 2, 3, \dots,$

$$\operatorname{Cov}\left(W_{t_i}^X, W_{t_j}^Y\right) = \rho \min(t_i, t_j), \text{ where } |\rho| < 1.$$

The existence of Wiener processes, which satisfy the Assumption $(\mathcal{C}_1)$ follows from standard stochastic calculus. Under Assumption $(\mathcal{C}_2)$ given below, the parameter ρ represents the correlation coefficient between $\log(X(t))$ and $\log(Y(t))$, for each $t \geqslant 0$.

The choice of a such error process with continuous paths is in agreement with the continuity in time of the solution of the ODEs (1). The error processes $\{e_t^X, t \geqslant 0\}$, $\{e_t^Y, t \geqslant 0\}$ are ergodic Markov processes, and under Assumption $(\mathcal{C}_2)$ given below, they are stationary Gaussian. Further, the model in (4) has as the advantage that it captures the irregularities, (i.e., roughness) and the periodic behaviour of paths observed in practice. For some examples of the models, where extinction is possible, see, e.g., Gard and Kannan [8], Gard [9] or Chen and Kulperger [5], and references therein.

For the the error processes to be stationary, we impose the following assumption.

Assumption $(\mathcal{C}_2)$**.** The random $(e_0^X, e_0^Y)'$ is independent of $\{(W_t^X, W_t^Y), t \geqslant 0\}$, and

$$(e_0^X, e_0^Y)' \sim \mathcal{N}_2(\mathbf{0}, \mathbf{\Theta}), \quad \text{where } \mathbf{\Theta} = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad \text{with} \quad \sigma^2 = \tau^2 / 2c.$$

The suggested approach requires the covariance matrix of $\log X(t)$, $\log Y(t), t \geqslant 0$; the processes of interest. This is obtained from the following proposition.

Proposition 2.1. *Let* (e_t^X, e_t^Y) *be a solution to the system* (5) *and assume that Assumptions* $(\mathcal{C}_1)$ *and* $(\mathcal{C}_2)$ *hold. Further, let* $\phi = \exp(-c)$. *Then,*

(i) $E\left(\begin{pmatrix} (e_t^X, & e_t^Y)' \\ (e_s^X, & e_s^Y)' \end{pmatrix}\right) = \mathbf{0}_{4\times 1},$ *and*

$$\mathrm{Var}\begin{pmatrix} (e_t^X, & e_t^Y)' \\ (e_s^X, & e_s^Y)' \end{pmatrix} = \begin{pmatrix} \mathbf{\Theta} & \mathbf{\Theta}\phi^{|t-s|} \\ \mathbf{\Theta}\phi^{|t-s|} & \mathbf{\Theta} \end{pmatrix};$$

(ii) *the vector* $((e_t^X, e_t^Y), (e_s^X, e_s^Y))'$ *is normal variate of mean* $\mathbf{0}$ *and covariance matrix defined in* (i). □

For simplicity and clarity, we consider to deal with realizations of these processes at discrete times $t_1, \ldots, t_N$ and take $t_1 = 1, \ldots, t_N = N$. Similar result can be proved for observations at any sequence $0 < t_1 < \cdots < t_N$ (see Nkurunziza [15], Corollary 1.6).

Corollary 2.2. *Let* (e_t^X, e_t^Y) *be a solution to the system* (5) *and suppose that Assumption* $(\mathcal{C}_1)$ *holds. Then, the process in discrete time* $\left\{\left(e_i^X, e_j^Y\right), i, j = 1, \ldots, n, \ldots\right\}$ *satisfies the equations*

$$e_i^X = \phi e_{i-1}^X + \varepsilon_i^X, \quad e_i^Y = \phi e_{i-1}^Y + \varepsilon_i^Y; \tag{6}$$

where the vector $(\varepsilon_i^X, \varepsilon_i^Y)'$ *follows a bivariate normal distribution of mean* 0 *and covariance matrix* $(1-\phi^2)\Theta$. *Moreover, for* $i \neq j$, *the random vector* $(\varepsilon_i^X, \varepsilon_i^Y)$ *is independent of* $\left(\varepsilon_j^X, \varepsilon_j^Y\right)$. □

The proof of Corollary 2.2 follows from standard stochastic calculus. Note that the statistical model, which is commonly used by ecologists, with population cycles is the linear autoregressive (AR) model (see Kendall et al. [11], Royama [17]), with an order less than or equal to 2. In our case, the periodicity is captured by the solution of the ODEs (1) and then, to simply some computations, we can reduce the order by considering an AR (1) model.

As a consequence to Proposition 2.1 and Corollary 2.2, below Corollary 2.3 gives the conditional expectation and the variance of $H(X_{n+1}, Y_{n+1})$ given the past. Let $\mathcal{F}_n$ denote the sigma field generated by $\{(X_j, Y_j), 0 \leq j < n\}$ and all null sets (with respect to Gaussian measure). Let $(x(n), y(n)) = (x_n, y_n)$, $(X(n), Y(n)) = (X_n, Y_n)$, let

$$P_{n,X} = \log(x_n) + \phi(\log(X_{n-1}) - \log(x_{n-1})),$$

$$P_{n,Y} = \log(y_n) + \phi(\log(Y_{n-1}) - \log(y_{n-1})). \tag{7}$$

Further, let

$$g_1(\sigma, \phi) = \exp[2\sigma^2(1-\phi^2)] - \exp[\sigma^2(1-\phi^2)],$$

$$g_2(\sigma, \phi, \rho) = \exp[\sigma^2(1-\phi^2)(1+\rho)] - \exp[\sigma^2(1-\phi^2)],$$

and let $g_3(\sigma, \phi) = 2\sigma^2(1-\phi^2)\exp[\sigma^2(1-\phi^2)/2]$.

Corollary 2.3 (Froda and Nkurunziza [7]). *If* $(\mathcal{C}_1)$-$(\mathcal{C}_2)$ *hold, then, for all* $n = 0, 1, \ldots$

(i) *The conditional expectation is*

$$E\left[H(X_{n+1}, Y_{n+1})|\mathcal{F}_n\right] = \left[\gamma e^{P_{n+1,X}} + \beta e^{P_{n+1,Y}}\right] e^{\sigma^2(1-\phi^2)/2}$$

$$- \left(\delta P_{n+1,X} + \alpha P_{n+1,Y}\right). \tag{8}$$

(ii) *The conditional variance is*

$$\begin{aligned}\operatorname{Var}\left[H(X_{n+1}, Y_{n+1})|\mathcal{F}_n\right] &= \left(\alpha^2 + \delta^2 + 2\alpha\delta\rho\right)\sigma^2(1-\phi^2)\\ &+ 2\gamma\beta^2 \exp(P_{n+1,X} + P_{n+1,Y}) g_2(\sigma, \phi, \rho)\\ &+ \left[\beta^2\gamma^2 \exp(2P_{n+1,X}) + \beta^2 \exp(2P_{n+1,Y})\right] g_1(\sigma, \phi, \rho)\\ &- \left\{(\gamma\beta\delta + \gamma\beta\alpha\rho)\exp(P_{n+1,X}) + (\alpha\beta + \delta\beta\rho)\exp(P_{n+1,Y})\right\} g_3(\sigma, \phi).\end{aligned} \tag{9}$$

(iii) *In particular, if* $\rho = 0$ *and* $\phi \to 0$ *with* σ *fixed, then, the process* $\{(e_t^X, e_t^Y), t \geqslant 0\}$ *converges (in law) to an i.i.d. process. Further, in the limit, we get the unconditional versions, i.e.,*

$$E\left[H(X_{n+1}, Y_{n+1})|\mathcal{F}_n\right] \xrightarrow[\phi\to 0]{a.s.} E\left[H(X_{n+1}, Y_{n+1})\right]$$

$$= e^{\sigma^2/2}(\gamma x_{n+1} + \beta y_{n+1}) - \delta \log(x_{n+1}) - \alpha \log(x_{n+1}),$$

$$\operatorname{Var}\left[H(X_{n+1}, Y_{n+1})|\mathcal{F}_n\right] \xrightarrow[\phi\to 0]{a.s.} \operatorname{Var}\left[H(X_{n+1}, Y_{n+1})\right]$$

with

$$\operatorname{Var}\left[H(X_{n+1}, Y_{n+1})\right] = \left[\beta^2\gamma^2 \exp(2\log(x_{n+1})) + \beta^2 \exp(2\log(y_{n+1}))\right] g_1(\sigma, 0)$$

$$- \left\{\gamma\beta\delta \exp(\log(x_{n+1})) + \alpha\beta \exp(\log(y_{n+1}))\right\} g_3(\sigma, 0) + \left(\alpha^2 + \delta^2\right)\sigma^2. \quad \square$$

The proof is similar to that given in Froda and Nkurunziza ([7], Proposition 2.5). This corollary is crucial in the estimation procedure.

3. Estimation

3.1. Some qualitative properties of our ODE

The fundamental result of our procedure is based on a re-parameterization of a certain class of ODE system. Namely, the following lemma gives a relationship between the trajectories $\psi_1(t), \psi_2(t), t \geqslant 0$ solutions of the following ODE systems:

$$\dot{\psi}_1(t) = f(\psi_1(t), \theta), \quad t \geqslant 0 \text{ with } \psi_1(0) = x_0 \text{ fixed, and} \tag{10}$$

$$\dot{\psi}_2(t) = \kappa(t) f(\psi_2(t), \theta), \quad t \geqslant 0 \text{ with } \psi_2(0) = x_0 \text{ fixed,} \tag{11}$$

where f is continuous function, κ is a positive and continuous function. Let

$$K(t) = \int_0^t \kappa(s) ds, \quad \text{for all } t \geqslant 0. \tag{12}$$

Lemma 3.1. *Assume that f is a continuous function that satisfies the uniqueness conditions of the solution of the ODE system* (10), *and assume that* (12) *holds. Further, let $\psi_1(t)$ and $\psi_2(t)$ be the solution of the ODE systems* (10) *and* (11), *respectively. Then,*

$$\psi_2(t) = \psi_1(K(t)), \quad \textit{for all } t \geqslant 0. \qquad \square$$

Corollaries 3.2 and 3.3 stated below follow directly from Lemma 3.1.

Corollary 3.2. *Assume that $(u(t), v(t))$ and $(x(t), y(t))$ are the solution of ODE systems* (1) *and* (2), *respectively. Also, let $\kappa(t)$ be a function that satisfies the conditions of Lemma* 3.1, *and let $K(t)$ be the function given by* (12). *Then, for all $t \geqslant 0$,*

$$x(t) = u(K(t)) \quad \textit{and} \quad y(t) = v(K(t)). \qquad \square \tag{13}$$

Corollary 3.3. *Assume that Corollary* 3.2 *holds. Then, the trajectories $(u(t), v(t))$ and $(x(t), y(t)), t \geqslant 0$ stay on the same closed curve, i.e., for all $t \geqslant 0$,*

$$H(x(t),\ y(t)) = H(u(t),\ v(t)) = \eta, \tag{14}$$

where H and η are given by (3). □

Note that Corollary 3.2 generalizes the first part of Lemma 1 given in Froda and Colavita [6] that plays a central role in Froda and Nkurunziza [7]. In the estimation method, one exploits the periodicity of the observed data. Under some conditions on the function $K(t)$ defined in (12), the period of $(x(t),\ y(t))$ is linear in ω, the period as $(u(t),\ v(t))$. The following assumption is a sufficient condition for a such relationship.

Assumption $(\mathcal{C}_3)$. The continuous and positive function $\kappa(t)$ satisfies (12) and for all $t \geqslant 0$,

$$K(t + ((\omega - b)/\lambda)) = K(t) + \omega \text{ with } \lambda > 0,\ 0 \leqslant b < \omega.$$

In order to illustrate the idea in Assumption $(\mathcal{C}_3)$, let us give some examples of function $\kappa(t)$, which satisfy this assumption. We have

$$\kappa(t) = \lambda > 0 \quad \text{or} \quad \kappa(t) = \frac{\lambda\omega}{\omega - b} + a\cos\left(\frac{2\pi\lambda t}{\omega - b}\right) \text{ with } 0 \leqslant b < \omega,\ |a| < \frac{\lambda\omega}{\omega - b}.$$

From Section 4, we consider the simple case, where $b = 0$. Further, under $(\mathcal{C}_3)$, we establish in the Appendix, the following proposition that gives a period of $(x(t),\ y(t))$.

Proposition 3.4. *Assume that $(\mathcal{C}_3)$ and the conditions of Lemma* 3.1 *hold, and suppose that $\psi_1(t)$ is a ω-periodic function. Then, $\psi_2(t)$ is a ϖ_d-periodic function with*

$$\varpi_d = (\omega - b)/\lambda. \quad \square \tag{15}$$

From Proposition 3.4, we establish Corollary 3.5.

Corollary 3.5. *Let $(x(t),\ y(t))$ be the solution of ODE system* (2), *and assume that Lemma* 3.1 *and Assumption $(\mathcal{C}_3)$ hold. Then, $(x(t),\ y(t))$ is ϖ_d-periodic function, where ϖ_d is given in* (15). □

Corollaries 3.2 and 3.5 generalize Lemma 1 of Froda and Colavita [6]. Indeed, taking $\kappa(t) = \lambda > 0,$ for all $t \geqslant 0,$ we get $K(t) = \lambda(t)$ that satisfies $(\mathcal{C}_3)$ with $b = 0.$

In the sequel, we consider the simple case, where $b = 0,$ which by Proposition 3.4 and Corollary 3.5 implies that $\varpi_d = \omega / \lambda.$ We note that ϖ_d corresponds to the observed period of data. In this paper, we assume that λ and $\varpi_d = \omega / \lambda$ are known and/or estimated from the previous studies. For the estimation of $\varpi_d,$ the reader is referred to, for example, Whittle [20], Spanjaard and White [18], Bulmer [4], and references therein. Moreover, the estimated value of λ can be obtained iteratively (taking $\lambda = 1$ as initial value), such that the mean predictor error is minimum.

The following corollary shows how to estimate the closed curve $\eta = H(x_0, y_0)$ based on the periods of the ODE systems (1) and (2).

Corollary 3.6. *Let* $(u_*(t), v_*(t))$ *be the solution of the ODE* (1), *where* $\gamma, \beta, \delta, \alpha$ *are replaced with* $(\gamma_*, \beta_*, \delta_*, \alpha_*) = (\gamma / |\eta|, \beta / |\eta|, \delta / |\eta|, \alpha / |\eta|),$ *where* $\eta \neq 0.$ *Then,* $(u_*(t), v_*(t))$ *is* ω_**-periodic function and under* $(\mathcal{C}_3),$ *we have*

$$\omega_* / (\lambda \varpi_d + b) = |\eta|, \tag{16}$$

where ϖ_d *is given in* (15). *In particular, if* $b = 0,$ *we have*

$$\omega_* / (\lambda \varpi_d) = |\eta|. \qquad \square \tag{17}$$

From Corollary 3.6, taking η as the right hand side of the closed curve (3), we first estimate $(\gamma_*, \beta_*, \delta_*, \alpha_*)$ that corresponds to the normalized closed curve. Further, we estimate ω_* and, taking ϖ_d as the observed period of data, we deduce an estimate of $|\eta|.$ Then, multiplying the estimates of $(\gamma_*, \beta_*, \delta_*, \alpha_*)$ and $|\eta|,$ we obtain an estimate of $(\gamma, \beta, \delta, \alpha).$

Let $\Lambda = (\gamma, \beta, \delta, \alpha)$ and $\Lambda_* = (\gamma / |\eta|, \beta / |\eta|, \delta / |\eta|, \alpha / |\eta|)$. The first two parameters in Λ or Λ_* are interaction parameters, while the last two are a death rate and a birth rate, respectively.

Below, we outline the main steps of the algorithm of the estimation procedure. For the detailed description of this procedure, we refer the reader to Froda and Nkurunziza ([7], Section 4). For simplicity, we assume only equally spaced times, i.e., $t_i = i$.

4. Algorithm

4.1. Main steps of algorithm

Step I-a) (estimation): We obtain preliminary estimates of the parameters $\gamma, \beta, \delta, \alpha,$ and of the solution $(x(t), y(t)), t > 0$. At this step, we ignore the dependence structure given in (4), as the i.i.d. case is a limit case of the present model (see Froda and Nkurunziza [7], Proposition 2.5). Thus, this step is essentially done as in Froda and Colavita [6], where the errors in (4) are taken i.i.d., and one solves an ordinary least squares (OLS) problem involving only $H(x, y)$. In particular, from relation (17), here we refine the estimation of η and accordingly, we refine the preliminary estimates of $\gamma, \beta, \delta, \alpha,$ and of the solution $(x(t), y(t)), t > 0$.

Let $\widehat{\gamma}^{(0)}, \widehat{\beta}^{(0)}, \widehat{\delta}^{(0)}, \widehat{\alpha}^{(0)}$ be these preliminary estimates, and solve the ODE system (2) with these estimated coefficients to obtain the preliminary estimates at times $n = 1, 2 \dots, N, (\widehat{x}^{(0)}(n), \widehat{y}^{(0)}(n))$. Note that, in Froda and Nkurunziza [7], at this step, instead of solving the ODE (2), one solve the ODE (1).

Step I-b) (prediction): From the estimates $\widehat{\gamma}^{(0)}, \widehat{\beta}^{(0)}, \widehat{\delta}^{(0)}, \widehat{\alpha}^{(0)},$ and $(\widehat{x}^{(0)}(n), \widehat{y}^{(0)}(n))$ obtained at Step I-a), we compute prediction errors

$$\widehat{e}_{n,X}^{(0)} = \log(X(n)) - \log(\widehat{x}^{(0)}(n)), \quad \widehat{e}_{n,Y}^{(0)} = \log(Y(n)) - \log(\widehat{y}^{(0)}(n)). \quad (18)$$

Then, we get the estimates of the noise parameters. Let these estimates be $\widehat{\phi}_0, \widehat{\sigma}_0^2, \widehat{\rho}_0$. Further, we compute preliminary predictions, by replacing the theoretical values in (7) with their estimates, let these predictions be $\widehat{P}_{n,X}^{(0)}$ and $\widehat{P}_{n,Y}^{(0)}$, $n = 1, 2, \ldots, N$.

Step II-a) (estimation): Using the preliminary predictions, $\widehat{P}_{n,X}^{(0)}$ and $\widehat{P}_{n,Y}^{(0)}$, and the preliminary estimates $(\widehat{x}^{(0)}(n), \widehat{y}^{(0)}(n))$ and $\widehat{\phi}_0, \widehat{\sigma}_0^2, \widehat{\rho}_0$ obtained in Step I, we re-estimate the four parameters $\gamma, \beta, \delta, \alpha$. This is done by least squares, where we take into account the dependence structure given in (4). Let $\widehat{\gamma}, \widehat{\beta}, \widehat{\delta}, \widehat{\alpha}$ be these (final) parameter estimates. Further, we solve (1), at times $K(t)$ for $\geqslant 0$, with the original parameters replaced with these final parameter estimates, and obtain $(\widehat{x}(t), \widehat{y}(t))$, $t > 0$.

Step II-b) (prediction): By using the final estimates $\widehat{\gamma}, \widehat{\beta}, \widehat{\delta}, \widehat{\alpha}$, and $(\widehat{x}(n), \widehat{y}(n))$, $n = 1, 2, \ldots, N$ obtained in Step II-a), we repeat Step I-b). Namely, we compute final estimates $\widehat{\phi}, \widehat{\sigma}^2, \widehat{\rho}$ from the new residuals,

$$\widehat{e}_{n,X} = \log(X(n)) - \log(\widehat{x}_n), \quad \widehat{e}_{n,Y} = \log(Y(n)) - \log(\widehat{y}(n)), \tag{19}$$

and thus, we obtain the final predictions $\widehat{P}_{n,X}$ and $\widehat{P}_{n,Y}$ from (7). □

In Subsection 4.2, we give some mathematical expressions of the steps of the algorithm. Once again, for more details, the reader is referred to Froda and Nkurunziza [7].

4.2. Closed curve and estimation criteria

As mentioned above, there exists a constant η such that any solution $(x(t), y(t))$ stays on the closed curve

$$H(x(t), y(t)) = \eta, \tag{20}$$

with $H(x(t), y(t)) = (\gamma x(t) - \delta \log x(t)) + (\beta y(t) - \alpha \log y(t))$, for all $t \geqslant 0$. If $\eta \neq 0$, let $(\gamma_*, \beta_*, \delta_*, \alpha_*) = (\gamma / |\eta|, \beta / |\eta|, \delta / |\eta|, \alpha / |\eta|)$ and let $(u_*(t), v_*(t))$ be the ODE (1), in which $(\gamma, \beta, \delta, \alpha)$ is replaced with

$(\gamma_*, \beta_*, \delta_*, \alpha_*)$. Also, let ω and ω_* be the period of $(u(t), v(t))$, and $(u_*(t), v_*(t))$, respectively. Then, by Corollary 3.6, we have

$$\frac{\omega_*}{\lambda \varpi_d} = |\eta|, \tag{21}$$

where here, η is the right hand side of the Equation (20). It should be noted that when $\lambda = 1$, the relation (21) corresponds to that given in Froda and Colavita [6] or in Froda and Nkurunziza [7]. Further, let

$$\Lambda = (\gamma, \beta, \delta, \alpha), \quad \Lambda_* = (\gamma_*, \beta_*, \delta_*, \alpha_*),$$

and let

$$H_*(x, y) = \frac{1}{|\eta|}\left\{e^{-\sigma^2/2}(\gamma x + \beta y) - (\delta \log x + \alpha \log y)\right\}.$$

For simplicity, we assume that $\eta > 0$, and then from Corollary 2.3, we have

$$E\left[H_*(X_{n+1}, Y_{n+1})\right] = \frac{1}{|\eta|} H(x_{n+1}, y_{n+1}) = \frac{\eta}{|\eta|},$$

that is, the unconditional expectation of $H_*(X_{n+1}, Y_{n+1})$ becomes 1.

Accordingly, as in Froda and Nkurunziza [7], the estimate of Λ_* can be derived from the least squares (LS) minimization

$$\sum_{n=0}^{N-1}\left\{H_*(X_{n+1}, Y_{n+1}) - 1\right\}^2, \tag{22}$$

where σ^2 is replaced by a preliminary estimate $\tilde{\sigma}_0^2$ (for more details, see Froda and Nkurunziza [7]).

For the asymptotic theory issue, we consider a slightly different least squares criteria, that is, relied to the periodicity of the data. The periodicity leads to the following grouping assumption as given in Froda and Nkurunziza [7].

Grouping assumption $(\mathcal{C}_4)$**.** Let $k \in \mathbb{N}$, $k > 4$ be fixed and, for $j = 1, 2, \ldots, k$, let $\mathcal{I}_j$ be the set of indices of the observations in group j; N_j is the cardinality of $\mathcal{I}_j$. Define $\mu_j > 0$ by

$$\frac{N_j}{N} = \mu_j > 0, \text{ with } \sum_{j=1}^{k} \mu_j = 1.$$

For each group $j = 1, 2, \ldots, k$ assume that data are such that

$$E[\log(X_i)] = \log(x^{(j)}), \quad E[\log(Y_i)] = \log(y^{(j)}), \ i \in \mathcal{I}_j, \ j = 1, 2, \ldots, k. \tag{23}$$

For such grouped data, we consider average versions of $H(X_i, Y_i, \Lambda)$ over the data in the same group, i.e., for $j = 1, 2, \ldots, k$,

$$\overline{H}^{(j)}(\Lambda) = \frac{1}{N_j} \sum_{i \in \mathcal{I}_j} H(X_i, Y_i, \Lambda), \quad \overline{H} = \left(\overline{H}^{(1)}, \overline{H}^{(2)}, \ldots, \overline{H}^{(k)}\right)'. \tag{24}$$

Correspondingly, let

$$H_*(X_i, Y_i, \Lambda_*) = e^{-\sigma^2/2}(\gamma_* X_i + \beta_* Y_i) - (\delta_* \log(X_i) + \alpha_* \log(Y_i)),$$

and consider the averages

$$\overline{H}_*^{(j)}(\Lambda_*) = \frac{1}{N_j} \sum_{i \in \mathcal{I}_j} H_*(X_i, Y_i, \Lambda_*), \quad \overline{H}_* = \left(\overline{H}_*^{(1)}, \overline{H}_*^{(2)}, \ldots, \overline{H}_*^{(k)}\right)'. \tag{25}$$

Finally, we consider the following weighted least square (WLS) problem: minimize in $\Lambda_* = (\gamma_*, \beta_*, \delta_*, \alpha_*)'$

$$\sum_{j=1}^{k} N_j \left(\widetilde{H}_*^{(j)}(\Lambda_*) - 1\right)^2, \tag{26}$$

where, in $\overline{H}_*^{(j)}(\Lambda_*)$ as given in (25), we replace σ^2 with a preliminary estimate, $\tilde{\sigma}_0^2$ and obtain $\widetilde{H}_*^{(j)}(\Lambda_*)$. It should be noted that, for the finite population case, by taking $N_j = 1$, $j = 1, 2, \ldots, k$, the criterion (22) and (26) are equivalent. Thus, in practice, one can apply (22) for simplicity.

Let $\widehat{\Lambda}_*$ be the estimate of Λ_* obtained from (26). Further, in order to obtain an estimate $\widehat{\Lambda}$, we use Corollary 3.6 (with $b = 0$) and set $|\widehat{\eta}| = \widehat{\omega}_* / \lambda\varpi_d$, where ϖ_d and $\widehat{\omega}_*$ are the period observed in the data and the period of the solution to the ODE system (1) of parameters $(\widehat{\gamma}_*, \widehat{\beta}_*, \widehat{\delta}_*, \widehat{\alpha}_*)$, respectively. To solve this system (and thus obtain its period), we need to estimate the initial value (x_0, y_0). As justified in Froda and Colavita [6], we take an initial value, $(\widehat{x}_0, \widehat{y}_0)$, on the estimated curve

$$(\widehat{\gamma}_* x - \widehat{\delta}_* \log x) + (\widehat{\beta}_* y - \widehat{\alpha}_* \log y) = 1. \tag{27}$$

Froda and Colavita [6] prove that $(\widehat{x}_0, \widehat{y}_0)$, which satisfies (27) is a strongly consistent estimator of (x_0, y_0), the initial value the ODE (2).

Then, we obtain a preliminary estimate Λ as

$$(\widehat{\gamma}_0, \widehat{\beta}_0, \widehat{\delta}_0, \widehat{\alpha}_0)' = \widehat{\Lambda}^{(0)} = \widehat{\Lambda}_* \cdot |\widehat{\eta}| = \widehat{\Lambda}_* \cdot \widehat{\omega}_* / (\lambda\varpi_d). \tag{28}$$

Since

$$(\gamma_* x - \delta_* \log x) + (\beta_* y - \alpha_* \log y) = 1 \Leftrightarrow (\gamma x - \delta \log x) + (\beta y - \alpha \log y) = \eta,$$

we can use the same estimated curve (27) to get an initial value for the system in $\gamma, \beta, \delta, \alpha$. Thus, at the end of this step, we solve the ODE system (2) with estimated coefficients $\widehat{\Lambda}^{(0)}$, initial value on the estimated closed curve (27), and we obtain the preliminary estimates of $(x(n), y(n))$ at times $n = 1, 2 \ldots, N$, $(\widehat{x}^{(0)}(n), \widehat{y}^{(0)}(n))$. Then, we obtain, $(\widehat{\sigma}_0^2, \widehat{\phi}_0, \widehat{\rho}_0)$, the preliminary estimates of the nuisance parameters (σ^2, ϕ, ρ), by using the same formulas as in Froda and Nkurunziza ([7], p. 421). For completeness, let

$$\bar{\widehat{e}}_X = \frac{1}{N}\sum_{n=1}^{N} \widehat{e}_{n,X}, \quad \bar{\widehat{e}}_Y = \frac{1}{N}\sum_{n=1}^{N} \widehat{e}_{n,Y},$$

and where $(\widehat{e}_{n,X}, \widehat{e}_{n,Y})$ is given by (19). Then, as a common estimator of σ^2, we take

$$\widehat{\sigma}_0^2 = \frac{1}{2N}\sum_{i=1}^{N}(\widehat{e}_{i,X} - \overline{\widehat{e}}_X)^2 + \frac{1}{2N}\sum_{i=1}^{N}(\widehat{e}_{i,Y} - \overline{\widehat{e}}_Y)^2. \tag{29}$$

Further, let $\mathbb{I}_A$ be an indicator function of the event A. In order to estimate ϕ and c, we take

$$\widehat{\phi}_0 = \frac{\sum_{i=0}^{N-1}\widehat{e}_{i+1,X}\widehat{e}_{i,X}}{2\sum_{i=0}^{N-1}(\widehat{e}_{i,X})^2} + \frac{\sum_{i=0}^{N-1}\widehat{e}_{i+1,Y}\widehat{e}_{i,Y}}{2\sum_{i=0}^{N-1}(\widehat{e}_{i,Y})^2}, \tag{30}$$

and

$$\widehat{c}_0 = -\log(\widehat{\phi}_0)\,\mathbb{I}_{\{\widehat{\phi}_0 > 0\}},$$

because $\phi = \exp(-c)$, for $c > 0$. Finally, we obtain an estimate of ρ,

$$\widehat{\rho}_0 = \frac{1}{\widehat{\sigma}_0^2(1 - \widehat{\phi}_0^2)}\frac{1}{N}\sum_{n=1}^{N}(\widehat{e}_{i,X} - \widehat{\phi}_0\widehat{e}_{i-1,X})(\widehat{e}_{i,Y} - \widehat{\phi}_0\widehat{e}_{i-1,Y}). \tag{31}$$

Further, from (7), we get the estimated predictions $\widehat{P}_{n,X}^{(0)}$ and $\widehat{P}_{n,Y}^{(0)}$. The rest of the steps of the algorithm are done as in Froda and Nkurunziza ([7], p. 423-p. 424).

Briefly, let $\widehat{h}_{n+1,n}^{(0)}$ and $\widehat{\mathrm{Var}}_n^{(0)}$ be the right sides of (8) and (9), respectively, with the parameters replaced by their corresponding preliminary estimates for $n = 1, 2, \ldots$. Also, let

$$\overline{h}^{(j)} = \frac{1}{N_j}\sum_{i \in I_j}\widehat{h}_{i+1,i}^{(0)} \quad \text{and} \quad \overline{\mathrm{Var}}^{(j)} = \frac{1}{N_j}\sum_{i \in I_j}\widehat{\mathrm{Var}}_i^{(0)}. \tag{32}$$

Then, by the Weighted Least Squares method, we find $\widehat{\Lambda}$, the final estimate of Λ. Namely, we minimize in Λ

$$\sum_{j=1}^{k} \frac{N_j \left\{\overline{H}^{(j)} - \overline{h}^{(j)}\right\}^2}{\overline{\mathrm{Var}}^{(j)}}. \tag{33}$$

Also, we solve (2) with the original parameters replaced with these final parameter estimates, and obtain $(\widehat{x}(n), \widehat{y}(n))$, $n = 1, 2, \ldots, N$. Further, in the similar way as in (29), (30), and (31), we get $(\widehat{\sigma}^2, \widehat{\phi}, \widehat{\rho})$, the final estimates of (σ^2, ϕ, ρ). Finally, from (7), we obtain the final prediction of population sizes.

5. Asymptotic Results

In this section, we prove that under the statistical model (4), if $(x(t), y(t))$ is the solution of (2), our estimator is strongly consistent, when the sample size N is large (see Proposition 5.1). Also, the data satisfy the equations in (4) with (2) (instead of (4) with (1)), we establish a result which shows that, asymptotically, our method refines the Froda and Nkurunziza [7] by $|1 - (1/\lambda)| \cdot 100\%$ (see Propositions 5.3 and 5.4). For the proofs of these propositions, we refer the reader to the Appendix.

Proposition 5.1. *Assume that* $(\mathcal{C}_1), (\mathcal{C}_2), (\mathcal{C}_3)$, *and* $(\mathcal{C}_4)$ *hold. Then*,

$$\widehat{\Lambda} \xrightarrow[N\to\infty]{a.s.} \Lambda, \text{ (where a.s. stands for “almost surely”)}. \qquad \square$$

Proposition 5.2. *Assume that* $(\mathcal{C}_1), (\mathcal{C}_2), (\mathcal{C}_3)$, *and* $(\mathcal{C}_4)$ *hold. Then*,

$$(\widehat{\sigma}^2, \widehat{\phi}, \widehat{\rho}) \xrightarrow[N\to\infty]{a.s.} (\sigma^2, \phi, \rho). \qquad \square$$

The proof is similar as given in Froda and Nkurunziza ([7], Proposition B.2.5).

In the sequel, we denote $\widehat{\Lambda}^{(0)}_{Fr-Nk}$, (where “$Fr - Nk$” stands for Froda and Nkurunziza) the preliminary estimator of Λ obtained from the Froda and Nkurunziza [7] method. Also, let $\widehat{\Lambda}_{Fr-Nk}$ be the final estimator obtained from the Froda and Nkurunziza [7] method. Recall that $\widehat{\Lambda}^{(0)}$ and $\widehat{\Lambda}$ denote, respectively, the preliminary and the final estimator of Λ obtained from the present method. For any 4-column vector $\boldsymbol{a} \in \mathbb{R}^4$, let $\boldsymbol{a}'$ denote the transpose of $\boldsymbol{a}$ and let $\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a}'\boldsymbol{a}}$ (the Euclidian norm in $\mathbb{R}^4$). We define the relative difference between $\widehat{\Lambda}^{(0)}$ and $\widehat{\Lambda}^{(0)}_{Fr-Nk}$ as

$$\vartheta^{(0)}(N) = \| \widehat{\Lambda}^{(0)} - \widehat{\Lambda}^{(0)}_{Fr-Nk} \| / \| \widehat{\Lambda}^{(0)}_{Fr-Nk} \|.$$

In other words, $\vartheta^{(0)}(N)$ measures how much the preliminary estimator $\widehat{\Lambda}^{(0)}$ corrects the preliminary estimator in Froda and Nkurunziza [7], in case $(x(t),\ y(y))$ satisfies the ODE in (2) (instead of the ODE in (1)).

Proposition 5.3. *Assume that* $(\mathcal{C}_3)$ *holds* (*with* $b = 0$). *Then*

$$\widehat{\Lambda}^{(0)} = \lambda^{-1}\widehat{\Lambda}^{(0)}_{Fr-Nk} \quad \textit{and} \quad \vartheta^{(0)}(N) = |1 - \lambda^{-1}| \textit{ for all } N \geqslant 4. \quad \square \tag{34}$$

In the similar way, we define the relative difference between $\widehat{\Lambda}$ and $\widehat{\Lambda}_{Fr-Nk}$ as

$$\vartheta(N) = \|\widehat{\Lambda} - \widehat{\Lambda}_{Fr-Nk}\| / \|\widehat{\Lambda}_{Fr-Nk}\|.$$

It measures how much the final estimator $\widehat{\Lambda}$ corrects the final estimator in Froda and Nkurunziza [7], in case $(x(t),\ y(y))$ satisfies the ODE in (2) (instead of the ODE in (1)). Proposition 5.4 proves that, when N is large, $\vartheta(N)$ tends to $|1 - (1/\lambda)| \cdot 100\%$.

Also, on one hand, from Propositions 5.3 and 5.4, it follows that, under the statistical model (4), if $(x(t),\ y(t))$ is the solution of the ODE (2) with $\lambda \neq 1$, the estimator given in Froda and Nkurunziza [7] is no longer consistent. On the other hand, when $\lambda = 1$, the presented estimator is equivalent to that given in Froda and Nkurunziza [7].

Proposition 5.4. *Assume that* $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, $(\mathcal{C}_4)$, *and* $(\mathcal{C}_3)$ *hold with* $(b = 0)$. *Then*,

$$\|\widehat{\Lambda}^{(0)} - \widehat{\Lambda}_{Fr-Nk}\| / \|\widehat{\Lambda}_{Fr-Nk}\| \xrightarrow[N\to\infty]{a.s.} |1 - \lambda^{-1}|, \quad and \quad \vartheta(N) \xrightarrow[N\to\infty]{a.s.} |1 - \lambda^{-1}|. \qquad \square \quad (35)$$

6. Data Analysis: Two Examples

6.1. Estimation and prediction

In this section, we illustrate the application of the suggested method on two real data sets. The first data set consists in 64 pairs of the mink-muskrat as given in Brockwell and Davis ([3], p. 557-558). For the mink-muskrat data set, the prey are the muskrat and the predator are the mink. Note that these data correspond to fur sales of the Hudson Bay Company, in the years 1848–1912. Also, it should be noted that, the same data has been used in Froda and Colavita [6], in Froda and Nkurunziza [7] as well as in Nkurunziza [15], to illustrate their estimation methods. Further, this data set has been studied by Bulmer [4], who observed an approximate ten-year cycle and commented on the fact that the muskrat cycle is due to predation by mink. Thus, the mink-muskrat is a predator-prey couple, which seems to satisfy the requirement that the muskrat (prey) is the main food supply for the mink (predator).

The second data set is the paramecium-didinium one as given in Luckinbill [14], and this consists of 62 paramecium caudatium and 62 didinium nasutum, which have been observed over 30 days; measured every half day. For this data set, the prey is paramecium caudatium, while the predator is didinium nasutum. This is a laboratory data set, where the predator feeds on one prey only. Also, some conditions of coexistence between the predator and the prey have been created by the experimenter. Namely, to sustain the prey population, both ciliates were placed in a special medium, which rendered the paramecium poorer food for the didinium. This in turn reduced the predator's efficiency and that allowed the coexistence with an approximate period of 4.5 days (or 9 half-days).

In order to illustrate our method, we take

$\kappa(t) = 1 + 0.5 \cos\left(\frac{2\pi}{\varpi_d} t\right)$ for both data sets, that is, here $\lambda = 1$ and thus, we are expecting to have equivalent results as in Froda and Nkurunziza [7].

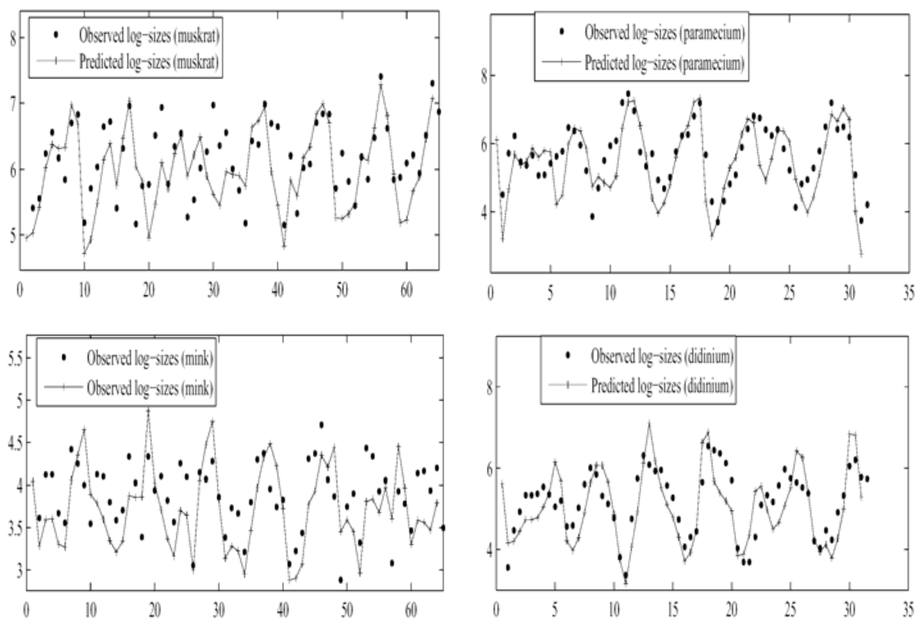
Applying our method, we obtain the parameter estimates, as given in Table 1. Moreover, the final estimates of the nuisance parameters are given in Table 2.

Table 1. Parameter estimates

Parameter	Mink-Muskrat Estimate of Λ		Didinium-Paramecium Estimate of Λ	
	Preliminary : $\widehat{\Lambda}^{(0)}$	Final : $\widehat{\Lambda}$	Preliminary : $\widehat{\Lambda}^{(0)}$	Final : $\widehat{\Lambda}$
γ	0.001001	0.001398	0.002911	0.007902
β	0.02288	0.023885	0.008921	0.011709
δ	0.509261	0.547321	1.333718	1.457290
α	0.976582	0.893325	1.96905	1.897283

Table 2. Parameter estimates

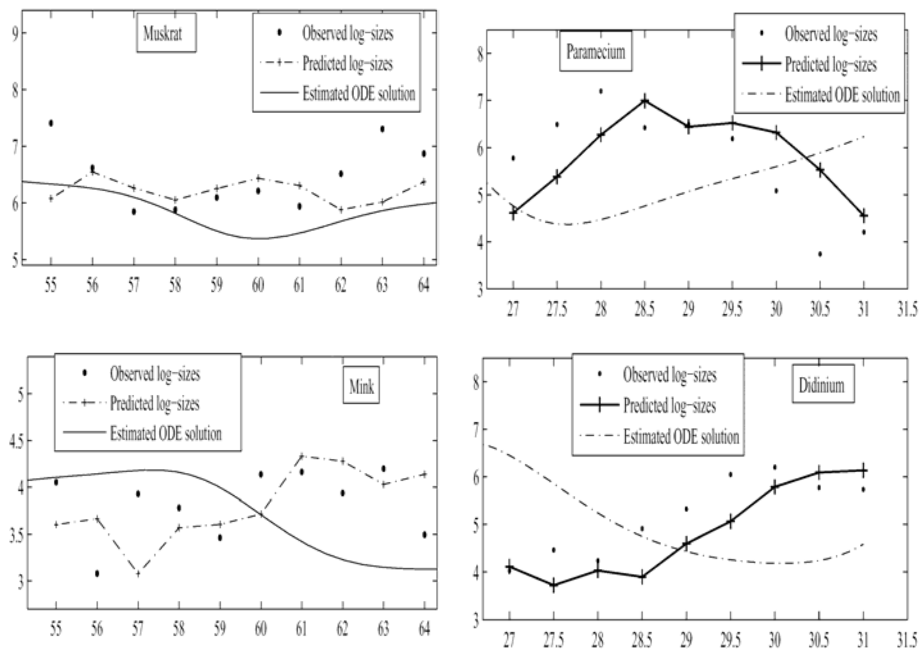
Parameter	Mink-Muskrat	Didinium-Paramecium
σ^2	0.5083	0.7380
ϕ	0.5612	0.7740
ρ	0.3393	0.4626



(a) (mink-muskrat) (b) paramecium-didinium

Figure 1. Observed and fitted logpopulation size.

Comparing our parameter estimates given in Table 1 to the parameter estimates given in Froda and Nkurunziza ([7], Table 1, p. 429), these numerical values confirm the theoretical result established in Section 5. In fact, our preliminary estimate corresponds to the preliminary estimate given the quoted paper. Furthermore, the relative difference, $\vartheta(N)$ is equal to 0.024193 and 0.015993, respectively, for the mink-muskrat and paramecium-didinium data sets and these numbers are relatively small (close to $|1 - 1/\lambda| = 0$).



(a) (mink-muskrat) (b) paramecium-didinium

Figure 2. Observed and fitted logpopulation size (out-of-sample).

In order to assess the performance of our estimators, we compare predicted population values, $\widehat{X}_n$ and $\widehat{Y}_n$, with their corresponding observed data points, X_n and Y_n. Graphically, Figures 1(a) and 1(b) highlight a good performance of our estimation method. Also, as we illustrate in the sequel, an out-of-sample prediction confirms this good performance of our estimation method.

As mentioned above, we use the out-of-sample prediction in order to evaluate the performance of our prediction model. For the mink-muskrat, we subtract the 10 last data points, while for the paramecium-didinium, we subtract 9 last data points. Then, by using the remaining data points (the first 54 and 53 for the mink-muskrat and paramecium-didinium, respectively), we estimate the parameters, and compute one-step-ahead forecasts corresponding to the out-of-sample data points. Comparing the one-step-ahead forecasts and their corresponding observed values, Figures 2(a) and 2(b) illustrate that the forecasts are quite goods.

7. Conclusion

In this paper, we consider a stochastic model that describes predator-prey interactions of two species, for which the coefficients of classical Lotka-Volterra ODE system are multiplied by a certain time-varying function. Further, we establish that the Froda and Nkurunziza [7] method is applicable to this more general case.

As a preliminary step, we establish a re-parameterization result that allows us to reduce a such ODE system to the classical Lotka-Volterra ODE system, whose parameters are constants and strictly positive. Finally, we give two illustrative examples by analyzing the Canadian mink-muskrat data set as well as a bacterial data set.

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Appendix

A. Some theoretical results

Proof of Proposition 3.4. Under $(\mathcal{C}_3)$,

$$K\left(t + \frac{\omega - b}{\lambda}\right) = K(t) + \omega, \quad \text{for all } t \geqslant 0.$$

Therefore, for all $t \geqslant 0$,

$$\psi_2\left(t + \frac{\omega - b}{\lambda}\right) = \psi_1(K(t) + \omega) = \psi_1(K(t)) = \psi_2(t),$$

that completes the proof. □

Proof of Proposition 5.1. The proof of the first statement is similar to that given in Froda and Nkurunziza ([7], Proposition B.1.5). Under $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, and $(\mathcal{C}_4)$, by Proposition B.1.4 given in Froda and

Nkurunziza [7], we have $\widehat{\Lambda}_* \xrightarrow[N\to\infty]{a.s.} \Lambda_*$. Further, in the similar way as in Froda and Colavita ([6], Proposition 6), when $N \to \infty$, $\widehat{\omega}_* / \lambda\varpi_d$ converges a.s. to $|\eta|$. Therefore,

$$\widehat{\Lambda}^{(0)} = \widehat{\Lambda}_* \frac{\widehat{\omega}_*}{\lambda\varpi_d} \xrightarrow[N\to\infty]{a.s.} \Lambda_*|\eta| = \Lambda.$$

Further, from Corollary 3.2, we have $(x(t), y(t)) = (u(K(t)), v(K(t)))$ and by Corollary 3.3, $(x(t), y(t))$ and $(u(t), v(t))$ stay on the same closed curve, that is, given in (20). Then, without loss of generality, one can take, $K(t) = t$, and hence, the rest of the proof is done as in Froda and Nkurunziza ([7], Proposition B.1.5). □

Proof of Proposition 5.3. From (28), we have $\widehat{\Lambda}^{(0)} = \widehat{\Lambda}_* \cdot \widehat{\omega}_* / \lambda\varpi_d$ and by Froda and Nkurunziza [7], $\widehat{\Lambda}^{(0)}_{Fr-Nk} = \widehat{\Lambda}_* \cdot \widehat{\omega}_* / \varpi_d$, that proves the first relation of (34).

Further, we get

$$\vartheta^{(0)}(N) = \frac{\|\widehat{\Lambda}^{(0)} - \widehat{\Lambda}^{(0)}_{Fr-Nk}\|}{\|\widehat{\Lambda}^{(0)}_{Fr-Nk}\|} = \frac{\|\frac{1}{\lambda}\widehat{\Lambda}^{(0)}_{Fr-Nk} - \widehat{\Lambda}^{(0)}_{Fr-Nk}\|}{\|\widehat{\Lambda}^{(0)}_{Fr-Nk}\|}$$

$$= \frac{|\frac{1}{\lambda} - 1|\,\|\widehat{\Lambda}^{(0)}_{Fr-Nk}\|}{\|\widehat{\Lambda}^{(0)}_{Fr-Nk}\|} = \left|1 - \frac{1}{\lambda}\right|,$$

that completes the proof. □

Proof of Proposition 5.4. From Proposition 5.3, we have

$$\|\widehat{\Lambda}^{(0)} - \widehat{\Lambda}^{(0)}_{Fr-Nk}\|^2 / \|\widehat{\Lambda}_{Fr-Nk}\|^2 = |1 - \lambda^{-1}|^2 \|\widehat{\Lambda}^{(0)}_{Fr-Nk}\|^2 / \|\widehat{\Lambda}_{Fr-Nk}\|^2,$$

and then, by using Propositions B.1.4 and B.1.5 in Froda and Nkurunziza [7], we get

$$\|\widehat{\Lambda}^{(0)}_{Fr-Nk}\|^2 / \|\widehat{\Lambda}_{Fr-Nk}\|^2 \xrightarrow[N\to\infty]{a.s.} 1, \quad \|\widehat{\Lambda}^{(0)}_{Fr-Nk} - \widehat{\Lambda}_{Fr-Nk}\|^2 / \|\widehat{\Lambda}_{Fr-Nk}\|^2 \xrightarrow[N\to\infty]{a.s.} 0, \tag{36}$$

and by some computations, we prove the first statement of the proposition. Further, one can verify that

$$\frac{\|\widehat{\Lambda} - \widehat{\Lambda}^{(0)}\|^2}{\|\widehat{\Lambda}_{Fr-Nk}\|^2} \xrightarrow[N \to \infty]{a.s.} 0,$$

and

$$\frac{(\widehat{\Lambda} - \widehat{\Lambda}^{(0)})'(\widehat{\Lambda}^{(0)} - \widehat{\Lambda}_{Fr-Nk})}{\|\widehat{\Lambda}_{Fr-Nk}\|^2} \xrightarrow[N \to \infty]{a.s.} 0,$$

(37)

and then,

$$\vartheta(N)^2 \xrightarrow[N \to \infty]{a.s.} |1 - (1 / \lambda)|^2,$$

and that completes the proof. □

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